Applied Numerical Mathematics 60 (2010) 1–9

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Bounds of the error of Gauss–Turán-type quadratures, II

Gradimir V. Milovanović [a,1], Miodrag M. Spalević [b,*,1], Miroslav S. Pranić [c,1]

[a] Megatrend University Belgrade, Faculty of Computer Sciences, Bulevar umetnosti 29, 11070 Novi Beograd, Serbia
[b] Department of Mathematics, University of Beograd, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Belgrade 35, Serbia
[c] Department of Mathematics and Informatics, University of Banja Luka, Faculty of Science, M. Stojanovića 2, 51000 Banja Luka, Bosnia and Herzegovina

ARTICLE INFO

Article history:
Received 4 September 2008
Received in revised form 2 August 2009
Accepted 8 August 2009
Available online 19 August 2009

MSC:
primary 65D30, 65D32
secondary 41A55

Keywords:
Gauss–Turán quadrature formula
Error bound
Remainder term for analytic functions
Contour integral representation

ABSTRACT

This paper is concerned with bounds on the remainder term of the Gauss–Turán quadrature formula,

$$R_{n,s}(f)=\int_{-1}^{1} f(t)w(t)\,dt-\sum_{\nu=1}^{n}\sum_{i=0}^{2s}\lambda_{i,\nu}f^{(i)}(\tau_\nu),$$

where

$$w(t)=w_{n,\ell}(t)=\big[U_{n-1}(t)/n\big]^{2\ell}\big(1-t^2\big)^{\ell-1/2}\quad(\ell\in\mathbb{N}),$$

U_{n-1} denotes the $(n-1)$th degree Chebyshev polynomial of the second kind and f is a function analytic in the interior of and continuous on the boundary of an ellipse with foci at ±1 and the sum of semi-axes $\varrho>1$.

1. Introduction

Let w be an integrable (nonnegative) weight function on the interval $(-1,1)$, $n\in\mathbb{N}$ and $s\in\mathbb{N}_0$. It is well known that Gauss–Turán quadrature formula with multiple nodes,

$$\int_{-1}^{1} f(t)w(t)\,dt=\sum_{\nu=1}^{n}\sum_{i=0}^{2s}\lambda_{i,\nu}f^{(i)}(\tau_\nu)+R_{n,s}(f),\tag{1.1}$$

is exact for all algebraic polynomials of degree at most $2(s+1)n-1$. The nodes τ_ν in (1.1) must be zeros of the corresponding s-orthogonal polynomials $\pi_n=\pi_{n,s}$ satisfying the following orthogonality conditions

$$\int_{-1}^{1}\pi_n(t)^{2s+1}t^k w(t)\,dt=0,\quad k=0,1,\dots,n-1.$$

Gauss–Turán quadrature formulae, or quadrature formulae with the highest degree of algebraic precision with multiple nodes, have extensively been studied in the last decades from both an algebraic and numerical point of view. Numerically

* Corresponding author.
E-mail addresses: gvm@megatrend.edu.rs (G.V. Milovanović), mspalevic@mas.bg.ac.rs (M.M. Spalević), pranic77m@yahoo.com (M.S. Pranić).
[1] The authors were supported in part by the Serbian Ministry of Science and Technological Development.

0168-9274/$30.00 © 2009 IMACS. Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.apnum.2009.08.002

stable methods for constructing nodes τ_ν and coefficients $\lambda_{i,\nu}$ can be found in [12] and [17]. Some interesting theoretical results concerning this theory have recently been obtained (see [16] (and references therein), [7,15]).

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and let D be its interior. If integrand f is analytic on D and continuous on $\overline{D}$, then the remainder term $R_{n,s}$ in (1.1) admits the contour integral representation

$$R_{n,s}(f) = \frac{1}{2\pi i}\oint_\Gamma K_{n,s}(z) f(z)\,dz. \tag{1.2}$$

The kernel is given by

$$K_{n,s}(z; w) = \frac{\rho_{n,s}(z; w)}{[\pi_{n,s}(z)]^{2s+1}}, \quad z \notin [-1,1], \tag{1.3}$$

where

$$\rho_{n,s}(z; w) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t} w(t)\,dt.$$

The modulus of the kernel is symmetric with respect to the real axis, i.e., $|K_{n,s}(\bar{z})| = |K_{n,s}(z)|$. If the weight function w is even, the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n,s}(-\bar{z})| = |K_{n,s}(z)|$ (see [8, Lemma 2.1]).

The integral representation (1.2) leads to a general error estimate, by using Hölder's inequality,

$$\left|R_{n,s}(f)\right| = \frac{1}{2\pi}\left|\oint_\Gamma K_{n,s}(z) f(z)\,dz\right| \leqslant \frac{1}{2\pi}\left(\oint_\Gamma \left|K_{n,s}(z)\right|^r |dz|\right)^{1/r} \left(\oint_\Gamma \left|f(z)\right|^{r'} |dz|\right)^{1/r'}, \tag{1.4}$$

i.e.,

$$\left|R_{n,s}(f)\right| \leqslant \frac{1}{2\pi}\|K_{n,s}\|_r \|f\|_{r'}, \tag{1.5}$$

where $1 \leqslant r \leqslant +\infty$, $1/r + 1/r' = 1$, and

$$\|f\|_r := \begin{cases} (\oint_\Gamma |f(z)|^r |dz|)^{1/r}, & 1 \leqslant r < +\infty, \\ \max_{z\in\Gamma} |f(z)|, & r = +\infty. \end{cases}$$

The case $r = +\infty$ $(r' = 1)$ gives

$$\left|R_{n,s}(f)\right| \leqslant \frac{1}{2\pi}\left(\max_{z\in\Gamma}\left|K_{n,s}(z)\right|\right)\|f\|_1, \tag{1.6}$$

whereas for $r = 1$ $(r' = +\infty)$ we have

$$\left|R_{n,s}(f)\right| \leqslant \frac{1}{2\pi}\left(\oint_\Gamma \left|K_{n,s}(z)\right| |dz|\right)\|f\|_\infty. \tag{1.7}$$

It is possible to obtain error bounds of the type (1.6) and (1.7) analytically (i.e., to calculate $\max_{z\in\Gamma}|K_{n,s}(z)|$ or $\oint_\Gamma |K_{n,s}(z)|\,|dz|$) only for weight functions which admit explicit Gauss–Turán quadrature formulae, i.e., in the cases when explicit formulae for corresponding s-orthogonal polynomials are known. There are only a couple of them.

In 1930, S. Bernstein [1] showed that the monic Chebyshev polynomial $\hat{T}_n(t) = T_n(t)/2^{n-1}$ minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}}\,dt \quad (k \geqslant 0).$$

This means that the Chebyshev polynomials T_n are s-orthogonal on $(-1,1)$ for each $s \geqslant 0$. Ossicini and Rosati [14] found three other weight functions $w_k(t)$ $(k = 2,3,4)$,

$$w_2(t) = \left(1-t^2\right)^{1/2+s}, \qquad w_3(t) = \frac{(1+t)^{1/2+s}}{(1-t)^{1/2}}, \qquad w_4(t) = \frac{(1-t)^{1/2+s}}{(1+t)^{1/2}},$$

for which the s-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third, and fourth kind: U_n, V_n, and W_n, which are defined by

$$U_n(\cos\theta)=\frac{\sin(n+1)\theta}{\sin\theta},\qquad V_n(\cos\theta)=\frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta},\qquad W_n(\cos\theta)=\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta},$$

respectively. These weight functions depend on s. It is easy to see that $W_n(-t)=(-1)^nV_n(t)$, so that in the investigation it is sufficient to study only the first three Jacobi measures $w_k(t)$, $k=1,2,3$.

Recently, Gori and Micchelli (see [4]) have introduced for each n the class of weight functions defined on $[-1,1]$ for which explicit Gauss–Turán quadrature formulae of all orders can be found. In other words, these weight functions do not depend on s, but depend on n. This class includes certain generalized Jacobi weight functions $w_{n,\mu}(t)=|U_{n-1}(t)/n|^{2\mu+1}(1-t^2)^\mu$, where $\mu>-1$. In this case, Chebyshev polynomials T_n appear as s-orthogonal polynomials.

Gauss–Turán quadratures with respect to the first four weight functions (including the case $s=0$) are considered in [2,3,8,13] (error bounds of the type (1.6)), [6,9] (error bounds of the type (1.7)).

Error bounds on Gauss–Turán quadratures with respect to the weight functions

$$w_{n,\mu}(t)=\left|U_{n-1}(t)/n\right|^{2\mu+1}\left(1-t^2\right)^\mu$$

are considered only in the particular case $\mu=1/2$ (cf. [10, Section 3; 11]). In this paper we consider more general case $\mu=\ell-1/2$, $\ell\in\mathbb{N}$, i.e.,

$$w(t)=w_{n,\ell}(t)=\frac{U_{n-1}^{2\ell}(t)}{n^{2\ell}}\left(1-t^2\right)^{\ell-1/2}\quad(\ell\in\mathbb{N}).\tag{1.8}$$

The paper is organized as follows. The explicit representation of the kernel of the remainder term in Gauss–Turán quadrature formulae with respect to the weight functions of the class (1.8) on elliptic contours is given in Section 2. Error bounds of the type (1.7), i.e., bounds on $\frac{1}{2\pi}\oint_\Gamma|K_{n,s}(z)|\,|dz|$ are derived in Section 3. Error bounds of the type (1.6), i.e., bounds on $\max_{z\in\Gamma}|K_{n,s}(z)|$ are derived in Section 4.

2. The modulus of the kernel on elliptic contours

Let contour Γ be an ellipse with foci at the points ±1 and a sum of semi-axes $\varrho>1$,

$$\mathcal{E}_\varrho=\left\{z\in\mathbb{C}\colon\ z=\frac{1}{2}\left(\xi+\xi^{-1}\right),\ \xi=\varrho e^{\mathrm{i}\theta},\ 0\leqslant\theta\leqslant2\pi\right\}.\tag{2.1}$$

As it is mentioned above we have that $\pi_{n,s}(t;w_{n,\ell})=T_n(t)$. We use the following facts (see [14, Eqs. (4.1) and (4.2)])

$$\left[T_n(t)\right]^{2s+1}=2^{-2s}\sum_{k=0}^{s}\binom{2s+1}{s-k}T_{n(2k+1)}(t),$$

$$\left(1-t^2\right)^s\left[U_n(t)\right]^{2s+1}=2^{-2s}\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s-k}U_{n(2k+1)+2k}(t),$$

and

$$\int_0^\pi\frac{\cos n\theta}{z-\cos\theta}\,d\theta=\frac{2\pi}{\xi-\xi^{-1}}\xi^{-n},\quad n\in\mathbb{N}_0,$$

$$\begin{aligned}
I_{j,p}&=\int_0^\pi\frac{\sin n\theta\,\sin(2p+1)n\theta\cos(2j+1)n\theta}{z-\cos\theta}\,d\theta\\
&=\frac{1}{4}\left[\int_0^\pi\frac{\cos(2p+2j+1)n\theta}{z-\cos\theta}\,d\theta+\int_0^\pi\frac{\cos(2j-2p+1)n\theta}{z-\cos\theta}\,d\theta\right.\\
&\qquad\left.-\int_0^\pi\frac{\cos(2p+2j+3)n\theta}{z-\cos\theta}\,d\theta-\int_0^\pi\frac{\cos(2j-2p-1)n\theta}{z-\cos\theta}\,d\theta\right]\\
&=\frac{1}{4}\frac{2\pi}{\xi-\xi^{-1}}\left[\frac{1}{\xi^{(2p+2j+1)n}}+\frac{1}{\xi^{|2j-2p+1|n}}-\frac{1}{\xi^{(2p+2j+3)n}}-\frac{1}{\xi^{|2j-2p-1|n}}\right]\\
&=\frac{\pi}{2}\frac{\xi^{2n}-1}{\xi^n\left(\xi-\xi^{-1}\right)}\left[\frac{1}{\xi^{2(j+p+1)n}}+\frac{\operatorname{sign}(p-j)}{\xi^{2|p-j|n}}\right].
\end{aligned}$$

Denoting $k = \ell - 1$, we get

$$\begin{aligned}
\rho_{n,s}(z; w_{n,\ell}) &= \int_{-1}^{1} \frac{[U_{n-1}(t)]^{2\ell}}{n^{2\ell}} \left(1-t^2\right)^{\ell-1} \sqrt{1-t^2}\, \frac{[T_n(t)]^{2s+1}}{z-t}\, dt \\
&= \int_{-1}^{1} \frac{U_{n-1}(t)\sqrt{1-t^2}}{n^{2\ell}} \frac{\{(1-t^2)^k [U_{n-1}(t)]^{2k+1}\}[T_n(t)]^{2s+1}}{z-t}\, dt \\
&= \int_{-1}^{1} \frac{U_{n-1}(t)\sqrt{1-t^2}}{n^{2\ell}(z-t)} \left[\frac{1}{2^{2k}} \sum_{p=0}^{k} (-1)^p \binom{2k+1}{k-p} U_{n(2p+1)-1}(t)\right]\left[\frac{1}{2^{2s}} \sum_{j=0}^{s} \binom{2s+1}{s-j} T_{n(2j+1)}(t)\right] dt.
\end{aligned}$$

By substituting $t = \cos\theta$, we have, in view of $T_n(\cos\theta) = \cos n\theta$ and $U_{n-1}(\cos\theta) = \sin n\theta / \sin\theta$,

$$\begin{aligned}
\rho_{n,s}(z; w_{n,\ell}) &= \frac{1}{n^{2\ell}4^{k+s}} \int_0^{\pi} \frac{\sin n\theta}{z-\cos\theta} \left[\sum_{p=0}^{k}\sum_{j=0}^{s} (-1)^p \binom{2k+1}{k-p}\binom{2s+1}{s-j} \frac{\sin(2p+1)n\theta}{\sin\theta} \cos(2j+1)n\theta\right] \sin\theta\, d\theta \\
&= \frac{1}{n^{2\ell}4^{k+s}} \sum_{p=0}^{k}\sum_{j=0}^{s} (-1)^p \binom{2k+1}{k-p}\binom{2s+1}{s-j} I_{j,p} \\
&= \frac{\pi}{2n^{2\ell}4^{k+s}} \frac{\xi^{2n}-1}{\xi^n(\xi-\xi^{-1})} \sum_{j=0}^{s}\sum_{p=0}^{k} (-1)^p \binom{2k+1}{k-p}\binom{2s+1}{s-j} \left[\frac{1}{\xi^{2(j+p+1)n}} + \frac{\operatorname{sign}(p-j)}{\xi^{2|j-p|n}}\right] \\
&= A_{n,s,k} \frac{\xi^{2n}-1}{\xi^{3n}(\xi-\xi^{-1})} V_{n,s,k}(\xi),
\end{aligned}$$

where

$$A_{n,s,k} = \frac{\pi}{2n^{2k+2}4^{k+s}},$$

$$V_{n,s,k}(\xi) = \sum_{\lambda=0}^{s+k} F_{s,k}(\lambda) \frac{1}{\xi^{2\lambda n}}, \tag{2.2}$$

and

$$F_{s,k}(\lambda) = \sum_{j+p=\lambda} (-1)^p \binom{2k+1}{k-p}\binom{2s+1}{s-j} + \sum_{|p-j|=\lambda+1} (-1)^p \operatorname{sign}(p-j) \binom{2k+1}{k-p}\binom{2s+1}{s-j}, \tag{2.3}$$

$j = 0, 1, \ldots, s;\ p = 0, 1, \ldots, k.$

According to (1.3) and well-known fact $T_n(z) = (\xi^n + \xi^{-n})/2$ we get

$$K_{n,s}(z; w_{n,\ell}) = B_{n,s,k} \frac{(\xi^n - \xi^{-n})}{\xi^{2n}(\xi-\xi^{-1})(\xi^n+\xi^{-n})^{2s+1}} V_{n,s,k}(\xi), \tag{2.4}$$

with $B_{n,s,k} = 2^{2s+1}A_{n,s,k} = \pi/(n^{2k+2}4^k)$. Further, using the well-known equalities

$$\left|\xi^n + \xi^{-n}\right| = \left[2(a_{2n} + \cos 2n\theta)\right]^{1/2}, \qquad \left|\xi^n - \xi^{-n}\right| = \left[2(a_{2n} - \cos 2n\theta)\right]^{1/2},$$

where

$$a_j = a_j(\varrho) = \frac{1}{2}\left(\varrho^j + \varrho^{-j}\right), \quad j \in \mathbb{N}, \tag{2.5}$$

we get an explicit representation of $K_{n,s}(z; w_{n,\ell})$ in the form

$$\left|K_{n,s}(z; w_{n,\ell})\right| = C_{n,s,k} \frac{(a_{2n} - \cos 2n\theta)^{1/2}}{(a_2 - \cos 2\theta)^{1/2}\,(a_{2n} + \cos 2n\theta)^{s+1/2}} \left|V_{n,s,k}(\xi)\right|, \tag{2.6}$$

where

$$C_{n,s,k} = \frac{\pi}{n^{2k+2}2^{s+2k+1/2}\varrho^{2n}},$$

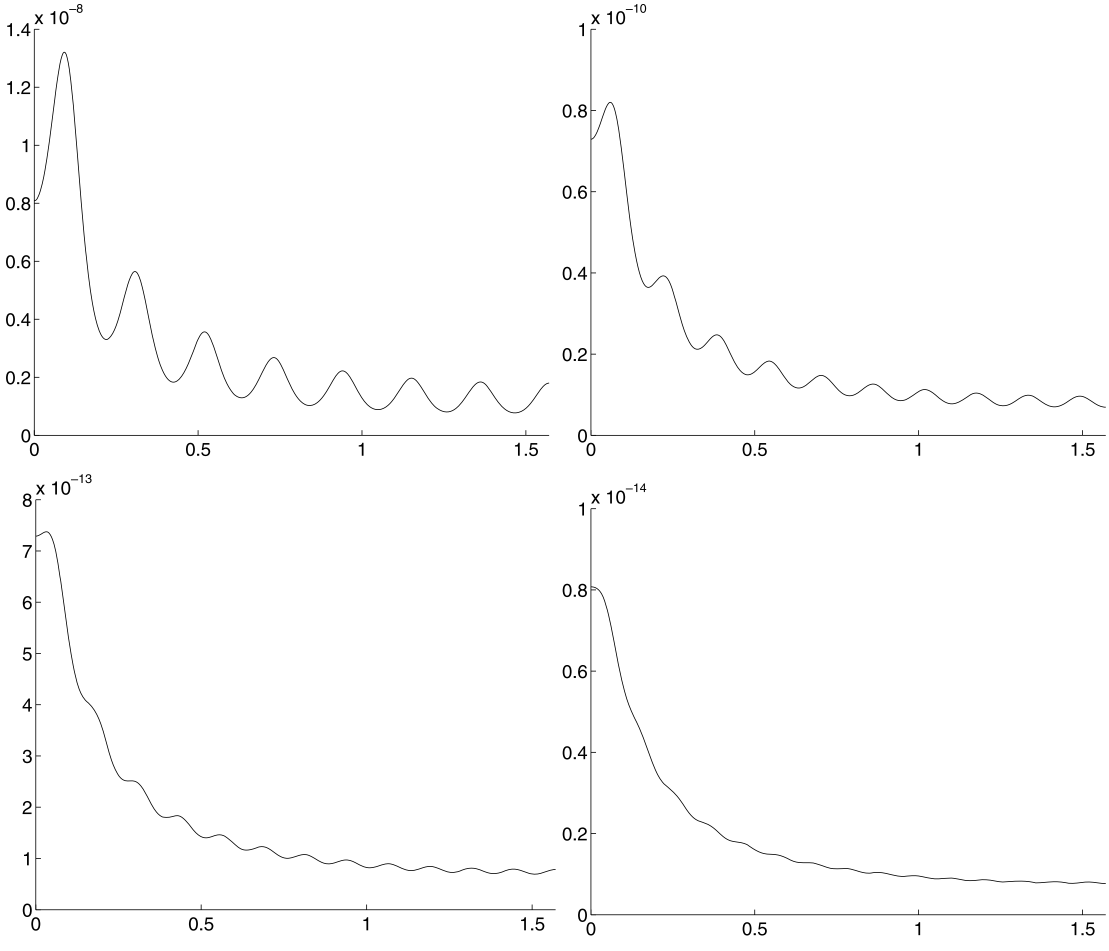

Fig. 1. The function $\theta \mapsto |K_{n,3}(z; w_{n,2})|$ $(z \in \mathcal{E}_{1.1})$ when $n = 15, 20, 25, 30$.

and (see [9, Lemma 4.1]),

$$\left|V_{n,s,k}\left(\varrho e^{i\theta}\right)\right| = \left[\varrho^{-2n(s+k)} \sum_{j=0}^{s+k} \bar{A}_j \cos 2jn\theta\right]^{1/2},$$

$$\bar{A}_0 = \frac{1}{x^{(s+k)/2}} \sum_{\nu=0}^{s+k} \left[F_{s,k}(s+k-\nu)\right]^2 x^{\nu}, \quad x = \varrho^{4n},$$

$$\bar{A}_j = \frac{2}{x^{(s+k-j)/2}} \sum_{\nu=0}^{s+k-j} F_{s,k}(s+k-\nu) F_{s,k}(s+k-\nu-j) x^{\nu}, \quad j = 1, \ldots, s. \tag{2.7}$$

Since $w_{n,\ell}$ is an even function, we have that $|K_{n,s}(z)|$, $z \in \mathcal{E}_\varrho$, is symmetric with respect to both axes. The graphs $\theta \mapsto |K_{n,s}(z; w_{n,\ell})|$ $(z \in \mathcal{E}_\varrho)$ for certain values of n, s, ℓ and ϱ are displayed in Fig. 1.

3. Error bounds of the type (1.7)

In this section we study the quantity $\frac{1}{2\pi} \oint_{\mathcal{E}_\varrho} |K_{n,s}(z; w_{n,\ell})| \, |dz|$, where integrand $|K_{n,s}(z; w_{n,\ell})|$ is given by (2.6). We use the following integral from [5, Eq. 3.616.7]

$$J_j(a) = \int_0^{\pi} \frac{\cos j\theta}{(a + \cos\theta)^{2s+1}} \, d\theta = \frac{(-1)^j \pi 2^{2s+1} x^{s-(j-1)/2}}{(x-1)^{4s+1}} \sum_{\nu=0}^{2s} \binom{2s+\nu}{\nu} \binom{2s+j}{\nu+j} (x-1)^{2s-\nu}, \tag{3.1}$$

where $a = (x+1)/(2\sqrt{x})$ and $x > 1$.

Theorem 3.1. *If a_j, $\bar{A}_j$ and J_j are defined by* (2.5), (2.7) *and* (3.1), *then we have*

$$\frac{1}{2\pi}\oint_{\mathcal{E}_\varrho}\left|K_{n,s}(z;w_{n,\ell})\right||dz| \leqslant \frac{\pi^{1/2}}{2^{s+2\ell-1}n^{2\ell}\varrho^{n(s+\ell+1)}} \times\left\{\sum_{j=0}^{s+\ell-1}\bar{A}_j\left[a_{2n}J_j(a_{2n})-\frac{1}{2}\big(J_{j+1}(a_{2n})+J_{|j-1|}(a_{2n})\big)\right]\right\}^{1/2}. \tag{3.2}$$

Proof. Since $z=\frac{1}{2}(\xi+\xi^{-1})$, $\xi=\varrho \mathrm{e}^{\mathrm{i}\theta}$, we have $|dz|=2^{-1/2}\sqrt{a_2-\cos 2\theta}\,d\theta$ and

$$\oint_{\mathcal{E}_\varrho}\left|K_{n,s}(z;w_{n,\ell})\right||dz| = D_{n,s,k}\int_0^{2\pi}\sqrt{\frac{a_{2n}-\cos 2n\theta}{(a_{2n}+\cos 2n\theta)^{2s+1}}\sum_{j=0}^{s+k}\bar{A}_j\cos 2jn\theta\,d\theta}, \tag{3.3}$$

where

$$D_{n,s,k}=\frac{C_{n,s,k}}{\sqrt{2}\varrho^{n(s+k)}}=\frac{\pi}{n^{2k+2}2^{s+2k+1}\varrho^{n(s+k+2)}}.$$

Using the fact

$$\int_0^{2\pi} g(2n\theta)\,d\theta = 2\int_0^{\pi} g(\theta)\,d\theta$$

and applying Cauchy's inequality, we get

$$\oint_{\mathcal{E}_\varrho}\left|K_{n,s,k}(z)\right||dz| \leqslant 2\sqrt{\pi}D_{n,s,k}\left(\int_0^{\pi}\frac{1}{(a_{2n}+\cos\theta)^{2s+1}}\sum_{j=0}^{s+k}\bar{A}_j[a_{2n}\cos j\theta-\cos\theta\cos j\theta]\,d\theta\right)^{1/2}$$

$$=2\sqrt{\pi}D_{n,s,k}\left\{\sum_{j=0}^{s+k}\bar{A}_j\left[a_{2n}J_j(a_{2n})-\frac{1}{2}\big(J_{j+1}(a_{2n})+J_{|j-1|}(a_{2n})\big)\right]\right\}^{1/2},$$

where

$$2\sqrt{\pi}D_{n,s,k}=\frac{\pi^{3/2}}{n^{2k+2}2^{s+2k}\varrho^{n(s+k+2)}}. \qquad \square$$

For $k=0$, the bound (3.2) coincides with the bound (2.10) from [11]. As is seen from Figs. 2 and 3, the bound (3.2) is very sharp, especially for larger values of n, s and ϱ.

Completing the examples corresponding to Figs. 2 and 3, we explicitly include the connected quadrature formulae. Namely, it is well known that the nodes in the quadrature formula (1.1), in the cases under consideration, are given by

$$\tau_\nu=\cos\frac{(2\nu-1)\pi}{2n}, \quad \nu=1,\dots,n.$$

The coefficients $\lambda_{i,\nu}$ are given by (see [18, Eq. (3.5)])

$$\lambda_{0,\nu}=\frac{\pi\varrho_0}{2n},$$

and, for $i=1,\dots,n$, by

$$\lambda_{i,\nu}=\frac{\pi}{2n}\sum_{j=[\frac{i+1}{2}]}^{s}\frac{(1-\tau_\nu^2)^j b_{2j-i,\nu,2j}}{(i-1)!2^{2j}jn^{2j}}\sum_{k=0}^{j}\binom{2j}{j-k}\varrho_k,$$

where

$$b_{k,\nu,j}=\frac{1}{k!}\big(L_\nu(t)^{-j}\big)^{(k)}_{t=\tau_\nu}, \quad k\in\mathbb{N}_0,\ \nu=1,\dots,n,\ j\in\mathbb{N},$$

$$L_\nu(t)=\frac{\omega_n(t)}{\omega_n'(\tau_\nu)(t-\tau_\nu)}, \quad \omega_n(t)=\prod_{i=1}^{n}(t-\tau_\nu),$$

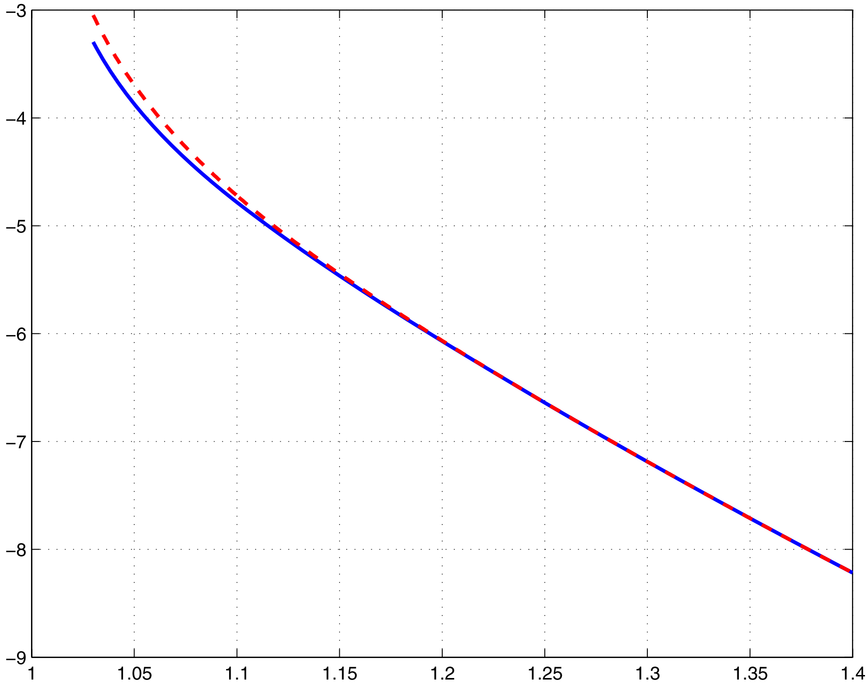

Fig. 2. Log_{10} of $1/(2\pi)\|K_{n,s}(z; w_{n,\ell})\|_1$ (solid line) and Log_{10} of the bound (3.2) (dashed line) as functions of ϱ, when $n = 8$, $s = 1$, $\ell = 2$.

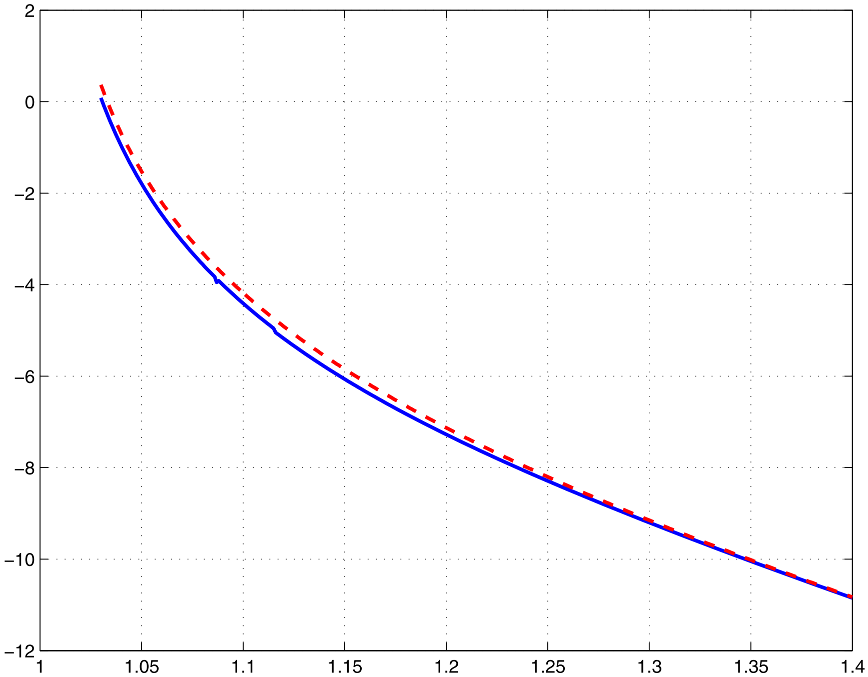

Fig. 3. Log_{10} of $1/(2\pi)\|K_{n,s}(z; w_{n,\ell})\|_1$ (solid line) and Log_{10} of the bound (3.2) (dashed line) as functions of ϱ, when $n = 5$, $s = 4$, $\ell = 3$.

Table 1
The coefficients $\lambda_{i,\nu}$ from (1.1) when $n=8$, $s=1$, $\ell=2$.

ν	$i=0$	$i=1$	$i=2$
1	3.59526747160697(−05)	4.59138725945918(−08)	1.78172812946410(−09)
2	3.59526747160697(−05)	3.89239017023565(−08)	1.44493370787881(−08)
3	3.59526747160697(−05)	2.60081196219861(−08)	3.23640414577609(−08)
4	3.59526747160697(−05)	9.13283709335839(−09)	4.50316504070851(−08)
5	3.59526747160697(−05)	−9.13283709335839(−09)	4.50316504070851(−08)
6	3.59526747160697(−05)	−2.60081196219861(−08)	3.23640414577609(−08)
7	3.59526747160697(−05)	−3.89239017023565(−08)	1.44493370787881(−08)
8	3.59526747160697(−05)	−4.59138725945918(−08)	1.78172812946410(−09)

Table 2
The coefficients $\lambda_{i,\nu}$ from (1.1) when $n=5$, $s=4$, $\ell=3$.

i	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=4$	$\nu=5$
0	1.25663706143592(−05)	$\lambda_{0,1}$	$\lambda_{0,1}$	$\lambda_{0,1}$	$\lambda_{0,1}$
1	3.37024752990368(−08)	2.08292752398086(−08)	0	$-\lambda_{1,2}$	$-\lambda_{1,1}$
2	3.48844458545855(−09)	2.31538143400258(−08)	3.53076812496826(−08)	$\lambda_{2,2}$	$\lambda_{2,1}$
3	2.36970846458994(−11)	9.90357058373093(−11)	0	$-\lambda_{3,2}$	$-\lambda_{3,1}$
4	4.90753754611256(−13)	1.85460459978200(−11)	4.26753946063638(−11)	$\lambda_{4,2}$	$\lambda_{4,1}$
5	3.58661614290293(−15)	9.82393244205083(−14)	0	$-\lambda_{5,2}$	$-\lambda_{5,1}$
6	3.32887129769973(−17)	7.43617507016751(−15)	2.58308729295161(−14)	$\lambda_{6,2}$	$\lambda_{6,1}$
7	1.44536614395942(−19)	2.87635125610980(−17)	0	$-\lambda_{7,2}$	$-\lambda_{7,1}$
8	5.18296467575908(−22)	1.14388006909791(−18)	6.23331875712263(−18)	$\lambda_{8,2}$	$\lambda_{8,1}$

Table 3
The values of ϱ_0 for certain values of n when $s=1$ and $\ell=3$.

n	ϱ_0	n	ϱ_0
3	2.24	7	1.38
4	1.79	8	1.33
5	1.58	9	1.29
6	1.46	10	1.26

and ϱ_k are the coefficients from Fourier–Chebyshev series of the form

$$w_{n,\ell}(t)\sqrt{1-t^2} = {\sum_{k=0}^{\infty}}' \varrho_k T_{2kn}(t),$$

where convergence holds with respect to the weighted L^1-norm

$$\int_{-1}^{1} |f(t)| \frac{dt}{\sqrt{1-t^2}}.$$

The prime on the summation indicates that the first term is halved.

The values of $\lambda_{i,\nu}$ corresponding to Figs. 2 and 3 are displayed in Tables 1 and 2.

4. Error bounds of the type (1.6)

In this section we study the quantity $\max_{z\in\mathcal{E}_\varrho} |K_{n,s}(z; w_{n,\ell})|$, where $|K_{n,s}(z)|$ is given by (2.6). Computation shows that $|K_{n,s}(z; w_{n,\ell})|$, $z \in \mathcal{E}_\varrho$, attains its maximum on the real axis ($z = \pm(\varrho + \varrho^{-1})/2$) if $\varrho > \varrho_0(n,s,\ell)$. Numerical values of ϱ_0 for certain values of n, s and ℓ have been determined by MATLAB and are shown in Tables 3 and 4. Displayed values are optimal in the sense that $|K_{n,s}(z; w_{n,\ell})|$ does not attain its maximum at $\theta = 0$ when $\varrho = \varrho_0 - 0.01$.

This empirical observation can be verified asymptotically as $\varrho \to \infty$. A lengthy calculation reveals that

$$\left|K_{n,s}(z; w_{n,\ell})\right| \sim \frac{\pi F_{s,k}(0)}{n^{2k+2} 4^k \varrho^{2n(s+1)+1}} \left(1 + \frac{2\cos 2\theta}{\varrho^2}\right)^{1/2}, \quad \varrho \to \infty.$$

Using the facts $|V_{n,s,k}(\xi)| \leqslant 4^{s+k}$ and $(a_2 - \cos 2\theta)(a_{2n} + \cos 2n\theta) \leqslant (a_2 - 1)(a_{2n} + 1)$ we obtain the following crude, but very simple inequality

$$\left|K_{n,s}(z; w_{n,\ell})\right| \leqslant \frac{\pi 4^s}{n^{2\ell} \varrho^{2n}} \frac{1}{(\varrho - \varrho^{-1})(\varrho^n - \varrho^{-n})^{2s}}. \tag{4.1}$$

Table 4
The values of ϱ_0 for certain values of ℓ when $s = 2$ and $n = 5$.

ℓ	ϱ_0	ℓ	ϱ_0
1	1.64	5	1.70
2	1.66	6	1.71
3	1.68	7	1.71
4	1.69	8	1.72

We conclude with some remarks about quadrature formulae studied here. In general, the nodes of Gauss–Turán quadrature formulae vary both with n and s, whereas in this case they are independent of s. This allows one to get higher precision by increasing s, without recalculating nodes. The convergence of (1.1) with respect to $w_{n,\ell}(t)$, when $s \to \infty$, immediately follows from (1.6) and (4.1). See also [4, Theorem 4.3].

References

[1] S. Bernstein, Sur les polynomes orthogonaux relatifs à un segment fini, J. Math. Pures Appl. (1930) 127–177.
[2] W. Gautschi, R.S. Varga, Error bounds for Gaussian quadrature of analytic functions, SIAM J. Numer. Anal. 20 (1983) 1170–1186.
[3] W. Gautschi, E. Tychopoulos, R.S. Varga, A note on the contour integral representation of the remainder term for Gauss–Chebyshev quadrature rule, SIAM J. Numer. Anal. 27 (1990) 219–224.
[4] L. Gori, C.A. Micchelli, On weight functions which admit explicit Gauss–Turán quadrature formulas, Math. Comp. 65 (1996) 1567–1581.
[5] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, 6th edn., Academic Press, San Diego, 2000 (A. Jeffrey, D. Zwillinger (Eds.)).
[6] D.B. Hunter, Some error expansions for Gaussian quadrature, BIT 35 (1995) 64–82.
[7] A. Kroó, F. Peherstorher, Asymptotic representation of L_p-minimal polynomials, $1 < p < \infty$, Constr. Approx. 25 (2007) 29–39.
[8] G.V. Milovanović, M.M. Spalević, Error bounds for Gauss–Turán quadrature formulae of analytic functions, Math. Comp. 72 (2003) 1855–1872.
[9] G.V. Milovanović, M.M. Spalević, An error expansion for Gauss–Turán quadratures and L^1-estimates of the remainder term, BIT 45 (2005) 117–136.
[10] G.V. Milovanović, M.M. Spalević, Bounds of the error of Gauss–Turán-type quadratures, J. Comput. Appl. Math. 178 (2005) 333–346.
[11] G.V. Milovanović, M.M. Spalević, A note on the bounds of the error of Gauss–Turán-type quadratures, J. Comput. Appl. Math. 200 (2007) 276–282.
[12] G.V. Milovanović, M.M. Spalević, A.S. Cvetković, Calculation of Gaussian type quadratures with multiple nodes, Math. Comput. Modell. 39 (2004) 325–347.
[13] G.V. Milovanović, M.M. Spalević, M.S. Pranić, Maximum of the modulus of kernels in Gauss–Turán quadratures, Math. Comp. 77 (2008) 985–994.
[14] A. Ossicini, F. Rosati, Funzioni caratteristiche nelle formule di quadratura gaussiane con nodi multipli, Bull. Un. Mat. Ital. 11 (1975) 224–237.
[15] F. Peherstorfer, Gauss–Turán quadrature formulas: Asymptotics of weights, SIAM J. Numer. Anal. 47 (2009) 2638–2659.
[16] Y.G. Shi, Christoffel type functions for m-orthogonal polynomials, J. Approx. Theory 137 (2005) 57–88.
[17] Y.G. Shi, G. Xu, Construction of σ-orthogonal polynomials and Gaussian quadrature formulas, Adv. Comput. Math. 27 (2007) 79–94.
[18] S. Yang, On a quadrature formula of Gori and Micchelli, J. Comput. Appl. Math. 176 (2005) 35–43.